

Operator monotonicity of some functions

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Abstract

We investigate the operator monotonicity of the following functions:

$$f(t) = t^\gamma \frac{(t^{\alpha_1} - 1)(t^{\alpha_2} - 1) \cdots (t^{\alpha_n} - 1)}{(t^{\beta_1} - 1)(t^{\beta_2} - 1) \cdots (t^{\beta_n} - 1)} \quad (t \in (0, \infty)),$$

where $\gamma \in \mathbb{R}$ and $\alpha_i, \beta_j > 0$ with $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$). This property for these functions has been considered by V.E. Szabó [11].

1 Introduction and Main results

We consider the following functions on $(0, \infty)$:

$$f(t) = \begin{cases} t^\gamma \prod_{i=1}^n (t^{\alpha_i} - 1) / (t^{\beta_i} - 1) & \text{if } t \neq 1 \\ \prod_{i=1}^n \alpha_i / \beta_i & \text{if } t = 1 \end{cases},$$

where $\gamma \in \mathbb{R}$ and $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$). These functions has been treated by V.E.S. Szabó and he has discussed their operator monotonicity in [11]. He gave a sufficient condition for that $f(t)$ becomes operator monotone on $(0, \infty)$. But his argument contained some errors and it is known the existence of a function $f(t)$ which satisfies this condition and is not operator monotone. We will make a new sufficient condition for that $f(t)$ becomes operator monotone on $(0, \infty)$.

Let $g(t)$ be a real valued continuous function on $(0, \infty)$. For a positive, invertible bounded linear operator A on a Hilbert space \mathcal{H} , we denote by $g(A)$ the continuous functional calculus of A by g . We call g operator monotone if $0 < A \leq B$ implies $g(A) \leq g(B)$.

We assume that g is not constant. We call g a Pick function on $(0, \infty)$ if $g(t)$ has an analytic continuation $g(z)$ to the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \Im z > 0\}$ and $g(z)$ maps \mathbb{H}_+ into \mathbb{H}_+ , where $\Im z$ means the imaginary part of z . It is known that a Pick function is operator monotone and conversely a non-constant operator monotone function is a Pick function (Löwner's theorem), see [1], [3],

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[4] and [6]. In this paper, we will show that some $f(t)$ is operator monotone by proving that $f(t)$ is a Pick function. For a continuous function g on $(0, \infty)$, g becomes operator monotone if there exists a sequence $\{g_n\}$ of operator monotone functions such that $\{g_n\}$ pointwise converges to g on $(0, \infty)$.

We assume that $f(t)$ is a Pick function. By definition, there exists a holomorphic function $f(z)$ on \mathbb{H}_+ with $f(\mathbb{H}_+) \subset \mathbb{H}_+$,

$$f(z) = z^\gamma \prod_{i=1}^n (z^{\alpha_i} - 1) / (z^{\beta_i} - 1) \quad (0 < \text{Arg} z < \pi)$$

and

$$\lim_{z \in \mathbb{H}_+ \rightarrow t} f(z) = f(t) \quad \text{for all } t > 0.$$

Since the imaginary part $\Im f(z)$ of $f(z)$ is harmonic and positive on \mathbb{H}_+ , $f(z)$ does not have a zero on \mathbb{H}_+ . This means that $|\gamma| \leq 2$ and $0 < \alpha_i \leq 2$ ($i = 1, 2, \dots, n$). We also have $0 < \beta_i \leq 2$ ($i = 1, 2, \dots, n$) because $f(z)$ does not have a singular point on \mathbb{H}_+ .

So we consider the problem that $f(t)$ becomes a Pick function on $(0, \infty)$ under the condition $|\gamma| < 2$, $0 < \alpha_i, \beta_i < 2$ and $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$). In this case, the function

$$f(z) = z^\gamma \prod_{i=1}^n (z^{\alpha_i} - 1) / (z^{\beta_i} - 1)$$

is holomorphic on \mathbb{H}_+ and continuous on the closure of \mathbb{H}_+ .

We denote by Arg the function from $\mathbb{C} \setminus \{0\}$ to $[0, 2\pi)$ with $z = |z|e^{i\text{Arg} z}$ for $z \in \mathbb{C} \setminus \{0\}$. When $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$, we define $\arg f(z)$ as follows:

$$\arg f(z) = \gamma \text{Arg} z + \sum_{i=1}^n (\text{Arg}(z^{\alpha_i} - 1) - \text{Arg}(z^{\beta_i} - 1)), \quad z \in \mathbb{H}_+.$$

Then we remark that, for $t > 0$,

$$\lim_{z \in \mathbb{H}_+ \rightarrow t} \arg f(z) = 0.$$

We may consider that $\arg f(z)$ is continuous on the closure $\overline{\mathbb{H}_+}$ of \mathbb{H}_+ except $\{0\}$ if $|\gamma| < 2$ and $0 < \alpha_i, \beta_i < 2$.

We define a function $F : [0, 2] \times [0, 2] \rightarrow \mathbb{R}$ as follows:

$$F(a, b) = \begin{cases} a - b & \text{if } a \geq b, 0 \leq b \leq 1 \\ a - 1 & \text{if } 1 < a, b \leq 2 \\ 0 & \text{if } a < b, 0 \leq a \leq 1 \end{cases}.$$

Then we can prove the following statement:

Theorem 1.1. Let $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$, $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$) and $\alpha_i \leq \alpha_j$, $\beta_i \leq \beta_j$ if $1 \leq i < j \leq n$. If it satisfies

$$0 \leq \gamma - \sum_{i=1}^n F(\beta_i, \alpha_i) \text{ and } \gamma + \sum_{i=1}^n F(\alpha_i, \beta_i) \leq 1,$$

then we have

$$f(t) = \begin{cases} t^\gamma \prod_{i=1}^n (t^{\alpha_i} - 1)/(t^{\beta_i} - 1) & \text{if } t \neq 1 \\ \prod_{i=1}^n \alpha_i / \beta_i & \text{if } t = 1 \end{cases}$$

is operator monotone on $(0, \infty)$.

This theorem implies the following statement as seen in Szabó's paper[11]: f is operator monotone if

$$0 \leq \gamma + \sum_{i=1}^u \alpha_i + (n-u) - (v + \sum_{j=v+1}^n \beta_j) \leq \gamma + u + \sum_{i=u+1}^n \alpha_i - \sum_{j=1}^v \beta_j - (n-v) \leq 1,$$

where $\alpha_1 \leq \dots \leq \alpha_u \leq 1 < \alpha_{u+1} \leq \dots \leq \alpha_n$ and $\beta_1 \leq \dots \leq \beta_v \leq 1 < \beta_{v+1} \leq \dots \leq \beta_n$. As examples we can show that

$$f_a(t) = a(1-a) \frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \quad (-1 \leq a \leq 2)$$

is an operator monotone function on $[0, \infty)$ and

$$c(\lambda, \mu) = (\lambda\mu)^{-1/2} \prod_{i=1}^n \frac{\sinh(\frac{\beta_i}{2} \log \frac{\lambda}{\mu})}{\sinh(\frac{\alpha_i}{2} \log \frac{\lambda}{\mu})}$$

is a Morozova-Chentsov function ([6],[7], [8] and [10]) if it satisfies

$$0 \leq \gamma + \sum_{i=1}^u \alpha_i + (n-u) - (v + \sum_{j=v+1}^n \beta_j) \leq \gamma + u + \sum_{i=u+1}^n \alpha_i - \sum_{j=1}^v \beta_j - (n-v) \leq 1,$$

where $\gamma = (1 + \sum_{i=1}^n (\alpha_i - \beta_i))/2$.

2 Proof of theorem

Lemma 2.1. Let $0 < b < a < 2$ and $z = re^{i\theta}$ ($0 \leq \theta = \text{Arg} z \leq \pi$). For any $\epsilon > 0$, there exists an $R > 0$ such that

$$r > R \Rightarrow \left| \arg \frac{z^a - 1}{z^b - 1} - (a-b)\theta \right| < \epsilon.$$

Proof. When $|z| \rightarrow \infty$, $(z^a - 1)/z^a = 1 - 1/z^a \rightarrow 1$ and $z^b/(z^b - 1) = 1 + 1/(z^b - 1) \rightarrow 1$. So we can choose an $R > 0$ such that $|\arg(z^a - 1)/z^a|, |\arg z^b/(z^b - 1)| < \epsilon/2$. Then we have

$$|\arg \frac{z^a - 1}{z^b - 1} - (a - b)\theta| = |\arg \frac{z^a - 1}{z^a} \frac{z^a}{z^b} \frac{z^b}{z^b - 1} - \arg \frac{z^a}{z^b}| < \epsilon.$$

□

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with

$$0 < \alpha_i, \beta_i < 2 \text{ and } \alpha_i \neq \beta_j \quad (i, j = 1, 2, \dots, n)$$

and set

$$g(z) = \begin{cases} \prod_{i=1}^n (z^{\alpha_i} - 1)/(z^{\beta_i} - 1) & \Im z \geq 0 \text{ and } z \neq 1 \\ \prod_{i=1}^n \alpha_i/\beta_i & z = 1 \end{cases}.$$

We define numbers $\theta(\alpha, \beta)$ and $\Theta(\alpha, \beta)$ as follows:

$$\begin{aligned} \theta(\alpha, \beta) &= \inf\{\arg g(re^{\pi i}) \mid 0 < r \leq 1\} \\ \text{and } \Theta(\alpha, \beta) &= \sup\{\arg g(re^{\pi i}) \mid 0 < r \leq 1\}. \end{aligned}$$

Since $\lim_{r \rightarrow 0+} \arg g(re^{\pi i}) = 0$, we have $\theta(\alpha, \beta) \leq 0$. Remarking the fact

$$\frac{(re^{\pi i})^a - 1}{(re^{\pi i})^b - 1} = (re^{\pi i})^{a-b} \frac{1 - (e^{-\pi i}/r)^a}{1 - (e^{-\pi i}/r)^b} = (re^{\pi i})^{a-b} \overline{\left(\frac{(e^{\pi i}/r)^a - 1}{(e^{\pi i}/r)^b - 1}\right)},$$

we can get the following relation:

$$\arg g(e^{\pi i}/r) + \arg g(re^{\pi i}) = \sum_{i=1}^n (\alpha_i - \beta_i)\pi.$$

Then the following numbers $F_0(\alpha, \beta)$ and $G_0(\alpha, \beta)$ can be represented by $\theta(\alpha, \beta)$ and $\Theta(\alpha, \beta)$ as follows:

$$\begin{aligned} F_0(\alpha, \beta) &= \sup\{\arg g(re^{\pi i})/\pi \mid r > 0\} \\ &= \max\{\Theta(\alpha, \beta), \sum_{i=1}^n (\alpha_i - \beta_i)\pi - \theta(\alpha, \beta)\}/\pi \end{aligned}$$

and

$$\begin{aligned} G_0(\alpha, \beta) &= \inf\{\arg g(re^{\pi i})/\pi \mid r > 0\} \\ &= \min\{\theta(\alpha, \beta), \sum_{i=1}^n (\alpha_i - \beta_i)\pi - \Theta(\alpha, \beta)\}/\pi. \end{aligned}$$

We enumerate the facts which follow from above observation:

$$\begin{aligned} \lim_{r \rightarrow 0+} \arg g(re^{\pi i}) &= 0, \quad \lim_{r \rightarrow \infty} \arg g(re^{\pi i}) = \sum_{i=1}^n (\alpha_i - \beta_i)\pi, \\ G_0(\alpha, \beta) &\leq \min\{0, \sum_{i=1}^n (\alpha_i - \beta_i)\}, \quad F_0(\alpha, \beta) \geq \max\{0, \sum_{i=1}^n (\alpha_i - \beta_i)\}, \\ G_0(\alpha, \beta) < 0 &\Leftrightarrow F_0(\alpha, \beta) > \sum_{i=1}^n (\alpha_i - \beta_i) \end{aligned}$$

and

$$G_0(\alpha, \beta) = -F_0(\beta, \alpha).$$

Theorem 2.2. *Let $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i < 2$, $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$) and*

$$f(t) = \begin{cases} t^\gamma \prod_{i=1}^n (t^{\alpha_i} - 1)/(t^{\beta_i} - 1) & t \in (0, \infty) \setminus \{1\} \\ \prod_{i=1}^n \alpha_i / \beta_i & t = 1 \end{cases}.$$

Then, for any $s > 0$, the following are equivalent:

- (1) $f(t)^s$ is operator monotone.
- (2) $\gamma - F_0(\beta, \alpha) \geq 0$ and $s(\gamma + F_0(\alpha, \beta)) \leq 1$.

Proof. (1) \Rightarrow (2) Since $f(t)^s$ is operator monotone, this implies

$$\begin{aligned} 0 &\leq \arg f(re^{\pi i})^s \leq \pi \\ \Rightarrow 0 &\leq s(\gamma\pi + \arg g(re^{\pi i})) \leq \pi \\ \Rightarrow 0 &\leq \gamma + G_0(\alpha, \beta) \text{ and } s(\gamma + F_0(\alpha, \beta)) \leq 1 \\ \Rightarrow 0 &\leq \gamma - F_0(\beta, \alpha) \text{ and } s(\gamma + F_0(\alpha, \beta)) \leq 1. \end{aligned}$$

(2) \Rightarrow (1) When $z = re^{\pi i}$ ($r > 0$), we have

$$s(\gamma - F_0(\beta, \alpha))\pi = s(\gamma + G_0(\alpha, \beta))\pi \leq \arg(f(z)^s) \leq s(\gamma + F_0(\alpha, \beta))\pi.$$

and, by assumption,

$$0 \leq \arg(f(z)^s) \leq \pi.$$

We remark that

$$0 \leq sF_0(\beta, \alpha) \leq s\gamma \leq 1 - sF_0(\alpha, \beta) \leq 1.$$

Since

$$\lim_{|z| \rightarrow 0} \arg g(z) = \lim_{|z| \rightarrow 0} \arg \left(\prod_{i=1}^n \frac{(z^{\alpha_i} - 1)}{(z^{\beta_i} - 1)} \right) = 0,$$

for any $\epsilon > 0$, we can choose $\delta_0 > 0$ such that $0 < |z| \leq \delta_0$ implies

$$|\arg f(z)^s - s\gamma \operatorname{Arg} z| < \epsilon.$$

In particular, we have $-\epsilon < \arg(f(z)^s) < \pi + \epsilon$ for any $z \in \mathbb{H}_+$ with $0 < |z| \leq \delta_0$.

To prove that $f(t)^s$ is operator monotone, it suffices to show that

$$0 \leq \arg(f(z)^s) \leq \pi \quad \text{for all } z \in \mathbb{H}_+.$$

We assume that there exists $z_0 \in \mathbb{H}_+$ such that $\arg(f(z_0)^s) < 0$ or $\arg(f(z_0)^s) > \pi$. For arbitrary $\epsilon > 0$, by Lemma 2.1, we can choose R_0 such that $R_0 > |z_0|$ and the condition $R > R_0$ implies

$$(\gamma + \sum_{i=1}^n (\alpha_i - \beta_i)) \operatorname{Arg} z - \epsilon < \arg f(z) < (\gamma + \sum_{i=1}^n (\alpha_i - \beta_i)) \operatorname{Arg} z + \epsilon$$

for all $z \in \mathbb{H}_+$ with $|z| = R$. When $\arg(f(z_0)^s) < 0$, we choose $\epsilon > 0$ with $\arg(f(z_0)^s) < \min\{-\epsilon, -s\epsilon\}$. It holds $|z_0| > \delta_0$ automatically, so we have $\arg(f(z_0)^s) < \arg(f(z)^s)$ if $z \in \mathbb{H}_+$ with $|z| = R$ or $|z| = \delta_0$. This contradicts to the maximum value principle of the harmonic function $\arg(f(\cdot)^s) (= s \arg(f(\cdot)))$ for the closed region $\{z \mid \Im z \geq 0, \delta_0 \leq |z| \leq R\}$. When $\arg(f(z_0)^s) > \pi$, we choose $\epsilon > 0$ with $\arg(f(z_0)^s) > \pi + s\epsilon$. Since $\sum_{i=1}^n (\alpha_i - \beta_i) \leq F_0(\alpha, \beta)$, we have $\arg(f(z_0)^s) > \arg(f(z)^s)$ if $z \in \mathbb{H}_+$ with $|z| = R$. This also implies a contradiction by the similar reason. \square

We put $s = 1$ and $\gamma = 0$ in the above proof of (2) \Rightarrow (1). Because $G_0(\alpha, \beta) \leq 0$ and $F_0(\alpha, \beta) \geq \sum_{i=1}^n (\alpha_i - \beta_i)$, we can get the following from this argument:

Corollary 2.3. *Let $0 < \alpha_i, \beta_i < 2$, $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$). Then we have*

$$-F_0(\beta, \alpha)\pi = G_0(\alpha, \beta)\pi \leq \arg g(z) \leq F_0(\alpha, \beta)\pi$$

for all z with $\Im z \geq 0$.

Moreover g is an operator monotone function on $[0, \infty)$ if and only if $0 \leq G_0(\alpha, \beta) \leq F_0(\alpha, \beta) \leq 1$.

Lemma 2.4. *Let $z = re^{i\pi}$ ($r > 0$) and $0 < b < a < 2$.*

(1) *If $b > 1$, then we have*

$$(1 - b)\pi \leq \arg \frac{z^a - 1}{z^b - 1} \leq (a - 1)\pi.$$

(2) *If $b \leq 1$, then we have*

$$0 \leq \arg \frac{z^a - 1}{z^b - 1} \leq (a - b)\pi.$$

Proof. (1) Since $\pi < \text{Arg}(z^a - 1) < \text{Arg}z^a = a\pi$ and $\pi < \text{Arg}(z^b - 1) < \text{Arg}z^b = b\pi$, we have

$$(1 - b)\pi < \arg \frac{z^a - 1}{z^b - 1} < (a - 1)\pi.$$

(2) When $b \leq 1 < a$, we have

$$0 < \arg \frac{z^a - 1}{z^b - 1} < (a - b)\pi,$$

since $\pi < \text{Arg}(z^a - 1) < \text{Arg}z^a = a\pi$ and $b\pi = \text{Arg}z^b \leq \text{Arg}(z^b - 1) \leq \pi$.

When $a \leq 1$, we consider the case that there exist positive integers k, l ($k > l$) such that $z^a = w^k$ (i.e., $w = r^{a/k} e^{a\pi i/k}$) and $z^b = w^l$. Since

$$\frac{z^a - 1}{z^b - 1} = \frac{w^k - 1}{w^l - 1} = z^{a-b} \cdot \frac{w^{k-1} + \dots + w + 1}{w^{k-l-1} + \dots + w^{k-l} + w^{k-1}}$$

and $\arg(w^{k-1} + w^{k-2} + \dots + w^m) \geq \arg(w^{m-1} + \dots + w + 1)$ for any $m = 1, 2, \dots, k - 1$, we have

$$0 \leq \arg \frac{z^a - 1}{z^b - 1} \leq (a - b)\pi.$$

By the continuity for a and b , it also holds for any a, b with $a \leq 1$. \square

Proposition 2.5. *Let $0 < a, b \leq 2$ and $z \in \mathbb{H}_+$. Then we have*

$$-F(b, a)\pi \leq \arg \frac{z^a - 1}{z^b - 1} \leq F(a, b)\pi,$$

where

$$F(a, b) = \begin{cases} a - b & \text{if } a \geq b, 0 \leq b \leq 1 \\ a - 1 & \text{if } 1 < a, b \leq 2 \\ 0 & \text{if } a < b, 0 \leq a \leq 1 \end{cases}.$$

Proof. We assume that $0 < b < a < 2$. Since $\arg \frac{z^a - 1}{z^b - 1} = -\arg \frac{z^b - 1}{z^a - 1}$, it suffices to show that

$$\begin{aligned} 0 &\leq \arg \frac{z^a - 1}{z^b - 1} \leq (a - b)\pi && \text{if } 0 < b \leq 1, \\ (1 - b)\pi &\leq \arg \frac{z^a - 1}{z^b - 1} \leq (a - 1)\pi && \text{if } 1 < b < 2. \end{aligned}$$

It is clear from Lemma 2.4 and Corollary 2.3.

We consider the case $a = 2$ or $b = 2$. Choose sequences $\{a_n\}, \{b_n\}$ with

$$0 < a_n, b_n < 2 \text{ and } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 2,$$

we have already shown the statement for $0 < a_n, b_n < 2$. Taking the limit, it holds for the case $a = 2$ or $b = 2$. \square

Theorem 2.6. Let $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$ and $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$). We denote by S_n the set of all permutations on $\{1, 2, \dots, n\}$. If it satisfies

$$0 \leq \gamma - \min_{\sigma \in S_n} \sum_{i=1}^n F(\beta_{\sigma(i)}, \alpha_i) \text{ and } \gamma + \min_{\sigma \in S_n} \sum_{i=1}^n F(\alpha_i, \beta_{\sigma(i)}) \leq 1,$$

then we have

$$f(t) = \begin{cases} t^\gamma \prod_{i=1}^n (t^{\alpha_i} - 1) / (t^{\beta_i} - 1) & \text{if } t \neq 1 \\ \prod_{i=1}^n \alpha_i / \beta_i & \text{if } t = 1 \end{cases}$$

is operator monotone on $(0, \infty)$.

Proof. We may assume that $0 < \alpha_i, \beta_i < 2$. Let $\sigma, \tau \in S_n$ with

$$\gamma - \sum_{i=1}^n F(\beta_{\sigma(i)}, \alpha_i) \geq 0, \quad \text{and} \quad \gamma + \sum_{i=1}^n F(\alpha_i, \beta_{\tau(i)}) \leq 1.$$

When $z = re^{i\pi}$ ($r > 0$), we have

$$\begin{aligned} -\sum_{i=1}^n F(\beta_{\sigma(i)}, \alpha_i) \pi &\leq \sum_{i=1}^n \arg \frac{z^{\alpha_i} - 1}{z^{\beta_{\sigma(i)}} - 1} = \arg \prod_{i=1}^n \frac{z^{\alpha_i} - 1}{z^{\beta_i} - 1} \\ &= \sum_{i=1}^n \arg \frac{z^{\alpha_i} - 1}{z^{\beta_{\tau(i)}} - 1} \leq \sum_{i=1}^n F(\alpha_i, \beta_{\tau(i)}) \pi \end{aligned}$$

by Proposition 2.5. This means that

$$-\sum_{i=1}^n F(\beta_{\sigma(i)}, \alpha_i) \leq G_0(\alpha, \beta) = -F_0(\beta, \alpha), F_0(\alpha, \beta) \leq \sum_{i=1}^n F(\alpha_i, \beta_{\tau(i)})$$

and

$$0 \leq \gamma - F_0(\beta, \alpha) \leq \gamma + F_0(\alpha, \beta) \leq 1.$$

So f is operator monotone by Theorem 2.2. \square

We remark that it holds

$$-\sum_{i=1}^n F(\beta_i, \alpha_i) \leq G_0(\alpha, \beta) \leq F_0(\alpha, \beta) \leq \sum_{i=1}^n F(\alpha_i, \beta_i)$$

from the argument in this proof.

For $0 \leq a_1 \leq a_2 \leq 2$ and $0 \leq b_1 \leq b_2 \leq 2$, we have

$$F(a_1, b_1) + F(a_2, b_2) \leq F(a_1, b_2) + F(a_2, b_1).$$

To prove this, we set $D = (F(a_1, b_2) + F(a_2, b_1)) - (F(a_1, b_1) + F(a_2, b_2))$ and show $D \geq 0$ in each of the following cases:

- | | |
|---|--|
| (1) $a_1 \geq 1$ | (2) $a_1 \leq 1 \leq a_2, b_1 \geq 1$ |
| (3) $a_1 \leq b_1 \leq 1 \leq a_2$ | (4) $b_1 \leq a_1 \leq 1 \leq a_2, 1 \leq b_2$ |
| (5) $b_1 \leq a_1 \leq 1 \leq a_2, a_1 \leq b_2 \leq 1$ | (6) $b_2 \leq a_1 \leq 1 \leq a_2$ |
| (7) $a_2 \leq 1, a_2 \leq b_1$ | (8) $a_2 \leq 1, a_1 \leq b_1 \leq a_2 \leq b_2$ |
| (9) $a_2 \leq 1, b_1 \leq a_1 \leq a_2 \leq b_2$ | (10) $a_1 \leq b_1 \leq b_2 \leq a_2 \leq 1$ |
| (11) $b_1 \leq a_1 \leq b_2 \leq a_2 \leq 1$ | (12) $b_2 \leq a_1, a_2 \leq 1$ |

Case (1). Since $(F(a_1, b_2) - F(a_1, b_1)) = (F(a_2, b_2) - F(a_2, b_1))$, $D = 0$.

Case (3). $D = (0 + (a_2 - b_1)) - (0 + F(a_2, b_2)) \geq 0$.

There are many easy calculations to show $D \geq 0$ in the rest cases (In particular, $D = 0$ in the cases (1), (2), (6), (7) and (12)). So we omit them.

By using the property

$$F(a_1, b_1) + F(a_2, b_2) \leq F(a_1, b_2) + F(a_2, b_1) \quad (a_1 \leq a_2, b_1 \leq b_2),$$

we can get the following statement:

Proposition 2.7. *Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in [0, 2]$ and $\sigma, \tau \in S_n$ permutations on $\{1, 2, \dots, n\}$ with*

$$a_{\sigma(i)} \leq a_{\sigma(j)}, b_{\tau(i)} \leq b_{\tau(j)} \quad \text{if } i < j.$$

Then we have

$$\sum_{i=1}^n F(a_{\sigma(i)}, b_{\tau(i)}) \leq \sum_{i=1}^n F(a_i, b_i).$$

Proof. We use an induction for n . In the case $n = 2$, we have already proved in above remark.

We assume that it holds for n , and will show that it also holds for $n + 1$. We may assume that

$$a_1 \leq a_2 \leq \dots \leq a_n \leq a_{n+1}.$$

We choose k with $b_k = \max\{b_1, b_2, \dots, b_{n+1}\}$. There is a permutation $\tau \in S_{n+1}$ such that

$$\tau(i) = \begin{cases} n+1 & i = k \\ k & i = n+1 \\ i & \text{otherwise} \end{cases}.$$

Since $a_k \leq a_{n+1}$ and $b_{n+1} \leq b_k$,

$$\begin{aligned} F(a_k, b_{\tau(k)}) + F(a_{n+1}, b_{\tau(n+1)}) &= F(a_k, b_{n+1}) + F(a_{n+1}, b_k) \\ &\leq F(a_k, b_k) + F(a_{n+1}, b_{n+1}). \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{i=1}^{n+1} F(a_i, b_i) &= \sum_{i \neq k, n+1} F(a_i, b_i) + F(a_k, b_k) + F(a_{n+1}, b_{n+1}) \\ &\geq \sum_{i \neq k, n+1} F(a_i, b_i) + F(a_k, b_{\tau(k)}) + F(a_{n+1}, b_{\tau(n+1)}) = \sum_{i=1}^{n+1} F(a_i, b_{\tau(i)}). \end{aligned}$$

By applying the hypothesis of induction for a_1, \dots, a_n and $b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_{n+1}$, there exists a permutation $\sigma \in S_{n+1}$ such that

$$\begin{aligned} \sigma(\{1, 2, \dots, n\}) &= \{1, 2, \dots, n+1\} \setminus \{k\}, \quad \sigma(n+1) = k \\ b_{\sigma(i)} &\leq b_{\sigma(j)} (\leq b_k) \quad \text{if } 1 \leq i < j \leq n \end{aligned}$$

and

$$\sum_{i=1}^n F(a_i, b_{\sigma(i)}) \leq \sum_{i=1}^n F(a_i, b_{\tau(i)}).$$

So it follows that

$$\begin{aligned} \sum_{i=1}^{n+1} F(a_i, b_{\sigma(i)}) &\leq \sum_{i=1}^n F(a_i, b_{\tau(i)}) + F(a_{n+1}, b_k) \\ &= \sum_{i=1}^n F(a_i, b_{\tau(i)}) + F(a_{n+1}, b_{\tau(n+1)}) \leq \sum_{i=1}^{n+1} F(a_i, b_i). \end{aligned}$$

□

We can see that Theorem 2.6 implies Theorem 1.1 and also that these two theorems are equivalent by Proposition 2.7.

In [11] Szabó remarked that

$$\begin{aligned} a\pi &\leq \text{Arg}(z^a - 1) \leq \pi & 0 \leq a \leq 1, \\ \pi &\leq \text{Arg}(z^a - 1) \leq a\pi & 1 \leq a \leq 2, \end{aligned}$$

when $z = re^{i\pi}$ ($r \geq 0$). This means that

$$-S(b, a)\pi \leq \arg \frac{z^a - 1}{z^b - 1} \leq S(a, b)\pi$$

for any $z = re^{i\pi}$, where

$$S(a, b) = \begin{cases} 1-b & 0 < a, b \leq 1 \\ 0 & 0 < a \leq 1 \leq b \leq 2 \\ a-b & 0 < b \leq 1 \leq a \leq 2 \\ a-1 & 1 \leq a, b \leq 2 \end{cases}.$$

Since $a - b \leq F_0(a, b) \leq F(a, b) \leq S(a, b)$, it follows from Theorem 1.1 that, for $|\gamma| \leq 2$ and $0 < \alpha_i, \beta_j \leq 2$ ($1 \leq i, j \leq n$) with $\alpha_1 \leq \dots \leq \alpha_u \leq 1 < \alpha_{u+1} \leq \dots \leq \alpha_n$ and $\beta_1 \leq \dots \leq \beta_v \leq 1 < \beta_{v+1} \leq \dots \leq \beta_n$,

$$f(t) = \begin{cases} t^\gamma \prod_{i=1}^n (t^{\alpha_i} - 1)/(t^{\beta_i} - 1) & \text{if } t \neq 1 \\ \prod_{i=1}^n \alpha_i / \beta_i & \text{if } t = 1 \end{cases}$$

is operator monotone on $[0, \infty)$ if

$$0 \leq \gamma - \sum_{i=1}^n S(b_i, a_i) \leq \gamma + \sum_{i=1}^n S(a_i, b_i) \leq 1.$$

In other words, $f(t)$ becomes operator monotone if

$$0 \leq \gamma + \sum_{i=1}^u \alpha_i + (n - u) - (v + \sum_{j=v+1}^n \beta_j) \leq \gamma + u + \sum_{i=u+1}^n \alpha_i - \sum_{j=1}^v \beta_j - (n - v) \leq 1.$$

When $\gamma = (1 + \sum_{i=1}^n (\alpha_i - \beta_i))/2$, it holds $f(t) = tf(1/t)$ and $c(\lambda, \mu) = \frac{1}{\mu f(\lambda \mu^{-1})}$ becomes Morozowa-Chentsov function ([11]).

3 Additional results

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with $\alpha_i, \beta_i > 0$ and $\alpha_i \neq \beta_j$ ($i, j = 1, 2, \dots, n$). We set

$$g(z) = \begin{cases} \prod_{i=1}^n (z^{\alpha_i} - 1)/(z^{\beta_i} - 1) & \Im z \geq 0 \text{ and } z \neq 1 \\ \prod_{i=1}^n \alpha_i / \beta_i & z = 1 \end{cases}.$$

Proposition 3.1. *If g is holomorphic on \mathbb{H}_+ and has no zeros on \mathbb{H}_+ , then $\max\{\alpha_i, \beta_j \mid i, j = 1, 2, \dots, n\} \leq 2$.*

In particular, if $\max\{\alpha_i, \beta_j \mid i, j = 1, 2, \dots, n\} > 2$, then $g(t)$ is not operator monotone on $[0, \infty)$.

Proof. In the case of $\alpha_1 > 2$. Then $e^{\pi i / \alpha_1} \in \mathbb{H}_+$ and $(e^{\pi i / \alpha_1})^{\alpha_1} - 1 = 0$. Because g has no zeros on \mathbb{H}_+ , there exists some β_j satisfying $(e^{\pi i / \alpha_1})^{\beta_j} - 1 = 0$, that is, $\beta_j \geq \alpha_1 > 2$.

In the case of $\beta_1 > 2$. Then $e^{\pi i / \beta_1} \in \mathbb{H}_+$ and $(e^{\pi i / \beta_1})^{\beta_1} - 1 = 0$. Since $g(z)$ is holomorphic on \mathbb{H}_+ , $e^{\pi i / \beta_1}$ is a removable singularity of $g(z)$. This means that $(e^{\pi i / \beta_1})^{\alpha_k} - 1 = 0$ for some α_k , that is, $\alpha_k \geq \beta_1 > 2$.

If $\max\{\alpha_i, \beta_j \mid i, j = 1, 2, \dots, n\} > 2$, then it contradicts to $\alpha_i \neq \beta_j$ for any $i, j = 1, 2, \dots, n$. \square

In the rest of this section, we assume that $0 \leq \alpha_i, \beta_i \leq 2$. We have already shown that

$$G_0(\alpha, \beta) \leq 0, \quad F_0(\alpha, \beta) \geq \sum_{i=1}^n (\alpha_i - \beta_i)$$

$$\text{and } G_0(\alpha, \beta) < 0 \Leftrightarrow F_0(\alpha, \beta) > \sum_{i=1}^n (\alpha_i - \beta_i),$$

where

$$F_0(\alpha, \beta) = \sup\{\arg g(re^{\pi i})/\pi \mid r > 0\} = \sup\{\arg g(z)/\pi \mid z \in \mathbb{H}_+\}$$

$$\text{and } G_0(\alpha, \beta) = -F_0(\beta, \alpha).$$

Proposition 3.2. *Let $0 < b < a \leq 2$. Then the following are equivalent:*

- (1) $F_0(a, b) = a - b$.
- (2) $G_0(a, b) = 0$.
- (3) $0 < b \leq 1$.

Proof. (1) \Leftrightarrow (2) It has been already stated in the above remark.

(3) \Rightarrow (2) Because $F(b, a) = 0$,

$$0 \geq G_0(a, b) = -F_0(b, a) \geq -F(b, a) = 0.$$

(2) \Rightarrow (3) It suffices to show that $G_0(a, b) < 0$ in the case $1 < b < 2$. We set $z = re^{i\theta}$ and

$$g(z) = \frac{z^a - 1}{z^b - 1} = \frac{1}{|z^b - 1|^2} (z^a - 1)(\bar{z}^b - 1)$$

$$= \frac{1}{|z^b - 1|^2} (r^{a+b} \cos(a-b)\theta - r^a \cos a\theta - r^b \cos b\theta + 1)$$

$$+ \frac{i}{|z^b - 1|^2} (r^{a+b} \sin(a-b)\theta - r^a \sin a\theta + r^b \sin b\theta).$$

When $1 < b < 2$, we choose a number θ satisfying

$$\frac{\pi}{b} < \theta < \pi.$$

For a sufficiently small positive r , we also have

$$\Re g(z) > 0 \text{ and } \Im g(z) < 0.$$

This means that $G_0(a, b) < 0$ if $1 < b < 2$. □

Example 3.3. (1) Let $\alpha_i > \beta_i$ and $\beta_i \leq 1$ ($i = 1, 2, \dots, n$). Then we have

$$F_0(\alpha, \beta) = \sum_{i=1}^n (\alpha_i - \beta_i) \text{ and } G_0(\alpha, \beta) = 0.$$

Moreover, $g(t) = \prod_{i=1}^n (t^{\alpha_i} - 1)/(t^{\beta_i} - 1)$ is operator monotone if and only if $0 \leq \sum_{i=1}^n (\alpha_i - \beta_i) \leq 1$.

- (2) For real numbers a, b with $|a|, |b| \leq 2$ and $a \neq b$, we define the function $h_1 : (0, \infty) \rightarrow \mathbb{R}$ as follows:

$$h_1(t) = \frac{b t^a - 1}{a t^b - 1},$$

where we consider $(t^a - 1)/a$ as $\log t$ in the case of $a = 0$. Then h_1 is operator monotone on $(0, \infty)$ if and only if

$$(a, b) \in \{(a, b) \in \mathbb{R}^2 \mid 0 < a - b \leq 1, a \geq -1, \text{ and } b \leq 1\} \\ \cup ([0, 1] \times [-1, 0]) \setminus \{(0, 0)\}.$$

- (3) For real numbers a, b with $a \neq b$, we define the function $h_2 : [0, \infty) \rightarrow \mathbb{R}$ as follows:

$$h_2(t) = \frac{t^a + 1}{t^b + 1}.$$

If h_2 is operator monotone on $[0, \infty)$, then $(a, b) \in [0, 1] \times [-1, 0]$. In particular, if $(0 < a - b \leq 1 \text{ and } b \leq 0 \leq a)$ or $(0 < a = -b \leq 1)$, then h_2 is operator monotone, and if $(a = 1 \text{ and } -1 < b < 0)$ or $(b = -1 \text{ and } 0 < a < 1)$, then h_2 is not operator monotone.

Proof. (1) By Proposition 3.2, we have

$$\sum_{i=1}^n (\alpha_i - \beta_i) = \sum_{i=1}^n F_0(\alpha_i, \beta_i) \geq F_0(\alpha, \beta) \geq \sum_{i=1}^n (\alpha_i - \beta_i).$$

This means that $F_0(\alpha, \beta) = \sum_{i=1}^n (\alpha_i - \beta_i)$ and $G_0(\alpha, \beta) = 0$.

The rest part follows from Corollary 2.3.

- (2) When $a, b \geq 0$, we have $\lim_{r \rightarrow \infty} \arg h_1(re^{i\pi}) = (a - b)\pi$. Since the operator monotonicity of h_1 implies $b < a$, we have that h_1 is operator monotone if and only if $0 < a - b \leq 1$ and $0 \leq b \leq 1$ by Proposition 3.2.

Assume that $a \geq 0 > b$. If h_1 is operator monotone, then we have $(a, b) \in [0, 1] \times [-1, 0]$ because

$$\lim_{r \rightarrow 0+} \arg h_1(re^{i\pi}) = -b\pi \text{ and } \lim_{r \rightarrow \infty} \arg h_1(re^{i\pi}) = a\pi.$$

The function h_1 can be written as follows:

$$h_1(t) = \frac{-b}{a} \cdot t^{-b} \cdot \frac{t^a - 1}{t^{-b} - 1}.$$

Since

$$-b - F(-b, a) = \begin{cases} -b & 0 \leq -b \leq a \leq 1 \\ a & 0 \leq a < -b \leq 1 \end{cases}$$

and

$$-b + F(a, -b) = \begin{cases} a & 0 \leq -b \leq a \leq 1 \\ -b & 0 \leq a < -b \leq 1 \end{cases},$$

h_1 is operator monotone if and only if $(a, b) \in [0, 1] \times [-1, 0]$ by Theorem 2.6.

Assume that $b > 0 \geq a$. Since $\lim_{r \rightarrow \infty} \arg h_1(re^{\pi i}) = -b\pi$, h_1 is not operator monotone.

Assume that $a, b \leq 0$. Since $\lim_{r \rightarrow 0+} \arg h_1(re^{\pi i}) = (a - b)\pi$, the operator monotonicity of h_1 implies $0 \leq a - b \leq 1$. We rewrite the function h_1 as follows:

$$h_1(t) = \frac{-b}{-a} \cdot t^{a-b} \cdot \frac{t^{-a} - 1}{t^{-b} - 1}.$$

We assume that $a < -1$ and $0 \leq a - b \leq 1$. Because $1 \leq -a \leq -b \leq 2$, we have $F_0(-b, -a) > (a - b)\pi$ by Proposition 3.2. So there exists $z_0 \in (-\infty, 0]$ such that

$$\arg \frac{z_0^{-a} - 1}{z_0^{-b} - 1} < (b - a)\pi \text{ and } \arg h_1(z_0) < 0,$$

that is, h_1 is not operator monotone. We assume that $a > -1$ and $0 \leq a - b \leq 1$. Since

$$a - b - F(-b, -a) = 0 \text{ and } a - b + F(-a, -b) = a - b \leq 1,$$

h_1 is operator monotone.

(3) Since

$$\begin{aligned} h_2(t) &= \frac{t^b - 1}{t^a - 1} \frac{t^{2a} - 1}{t^{2b} - 1} = t^{-b} \frac{t^{-b} - 1}{t^a - 1} \frac{t^{2a} - 1}{t^{-2b} - 1} \\ &= t^a \frac{t^b - 1}{t^{-a} - 1} \frac{t^{-2a} - 1}{t^{2b} - 1} = t^{a-b} \frac{t^{-b} - 1}{t^{-a} - 1} \frac{t^{-2a} - 1}{t^{-2b} - 1}, \end{aligned}$$

h_2 is not operator monotone if $(a, b) \notin [-1, 1] \times [-1, 1]$ by Proposition 3.1.

For $z = re^{i\pi}$, we have

$$\begin{aligned} h_2(z) &= \frac{1}{|z^b + 1|^2} (z^a + 1)(\bar{z}^b + 1) \\ &= \frac{1}{|z^b + 1|^2} (r^{a+b} \cos(a - b)\pi + r^a \cos a\pi + r^b \cos b\pi + 1) \\ &\quad + \frac{i}{|z^b + 1|^2} (r^{a+b} \sin(a - b)\pi + r^a \sin a\pi - r^b \sin b\pi). \end{aligned}$$

We assume $0 \leq a, b \leq 1$. Since $\lim_{r \rightarrow \infty} \arg h_2(re^{\pi i}) = (a - b)\pi$, h_2 is not operator monotone when $b > a$. If $0 < b < a \leq 1$, then there exists a sufficiently small $r > 0$ such that

$$\Im h_2(re^{\pi i}) = \frac{r^b}{|z^b + 1|^2} (r^a \sin(a - b)\pi + r^{a-b} \sin a\pi - \sin b\pi) < 0,$$

that is, h_2 is not operator monotone.

We assume $-1 \leq a, b < 0$. Applying the similar argument for the facts

$$\lim_{r \rightarrow 0+} \arg h_2(re^{\pi i}) = (a - b)\pi$$

and, for a sufficiently large $r > 0$, $\Im h_2(re^{\pi i}) < 0$ when $-1 \leq b < a < 0$, we can show that h_2 is not operator monotone.

We assume $-1 \leq a \leq 0$ and $0 < b \leq 1$. Since $\lim_{r \rightarrow \infty} \arg h_2(re^{\pi i}) = -b\pi < 0$, h_2 is also not operator monotone.

We assume $0 \leq a \leq 1$ and $-1 \leq b \leq 0$. We can easily show that, for $z = re^{i\theta}$ ($0 \leq \theta \leq \pi$), $\{\arg h_2(z)\}$ uniformly converges to $a\theta$ (resp. $-b\theta$) when r tends to ∞ (resp. r tends to 0). If $0 \leq a - b \leq 1$, then we have $\Im h_2(re^{\pi i}) \geq 0$ because $\sin(a - b)\pi, \sin a\pi, -\sin b\pi \geq 0$. Using the same method as the proof of Theorem 2.2, we can get the operator monotonicity of h_2 . If $0 < a = -b \leq 1$, then

$$\Im h_2(re^{\pi i}) = \frac{1}{|z^{-a} + 1|^2} (2 \cos a\pi + r^a + r^{-a}) \sin a\pi \geq 0.$$

So is h_2 .

If $a = 1$ and $-1 < b < 0$, then we have

$$\Im h_2(re^{\pi i}) = \frac{1}{|z^b + 1|^2} r^b (r - 1) \sin b\pi < 0$$

for some $r > 0$. If $b = -1$ and $0 < a < 1$, then we have

$$\Im h_2(re^{\pi i}) = \frac{1}{|z^{-1} + 1|^2} r^a (1 - \frac{1}{r}) \sin a\pi < 0$$

for some $r > 0$. So, for these 2 cases, h_2 is not operator monotone. \square

We consider the function h_1 (resp. h_2) as an extension of the representing function M_α (resp. L_p) of the power difference mean (resp. the Lehmer mean), where

$$M_\alpha(t) = \frac{\alpha - 1}{\alpha} \frac{t^\alpha - 1}{t^{\alpha-1} - 1}, \quad (-1 \leq \alpha \leq 2)$$

and

$$L_p(t) = \frac{t^p + 1}{t^{p-1} + 1}, \quad (0 \leq p \leq 1)$$

(see [5], [9]).

References

- [1] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
- [2] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, 2007.
- [3] W. F. Donoghue, Jr., *Monotone matrix functions and analytic continuation*, Springer-Verlag, 1974.
- [4] F. Hiai, *Matrix Analysis: Matrix monotone functions, matrix means, and majorization*, Interdecip. Inform. Sci. 16 (2010) 139–248.
- [5] F. Hiai and H. Kosaki, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J. 48 (1999), 899–936.
- [6] F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications*, Universitext, Springer, 2014.
- [7] W. Kumagai, *A characterization of extended monotone metrics*, Linear Algebra Appl. 434(2011), 224–231.
- [8] E.A. Morozova and N. N. Chentsov, *Markov invariant geometry on state manifolds*, Itogi Nauki i Techniki 36 (1990), 69–102.
- [9] Y. Nakamura, *Classes of operator monotone functions and Stieltjes functions*, The Gohberg anniversary collection, Vol. II, 395–404, Oper. Theory Adv. Appl., 41, Birkhäuser, Basel, 1989.
- [10] D. Petz, *Monotone metrics on matrix spaces*, Linear Algebra Appl. 244(1996), 81–96.
- [11] V.E.S. Szabó, *A class of matrix monotone functions*, Linear Algebra Appl. 420(2007), 79–85.